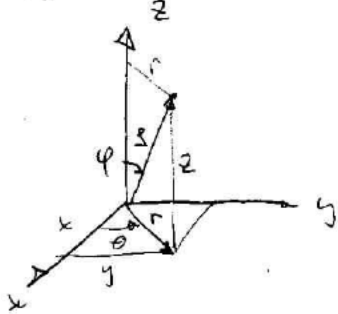
10.7

(4)

Spherical Coord



$r = \rho \sin\varphi$
$z = \rho \cos\varphi$

$x = r\cos\theta = \rho\sin\varphi \cdot \cos\theta$
$y = r\sin\theta = \rho\sin\varphi \; \sin\theta$

$$\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$$

$x = \rho\sin\varphi\cos\theta$
$y = \rho\sin\varphi\sin\theta$
$z = \rho\cos\varphi$

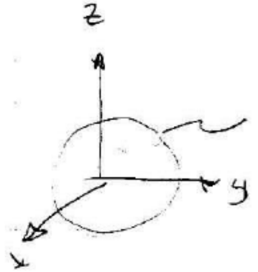
$\tan\theta = \frac{x}{y} \Rightarrow \theta = \tan^{-1}(x/y)$

Then
$r = \frac{x}{\cos\theta}$ or $\frac{y}{\sin\theta}$

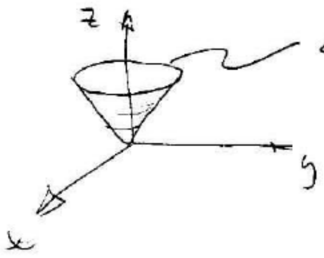
then $\rho = \sqrt{r^2+z^2}$

$$\varphi = \sin^{-1}\left(x/\rho\cos\theta\right)$$
$$= \sin^{-1}\left(y/\rho\sin\theta\right)$$
$$= \cos^{-1}\left(z/\rho\right)$$

Iso-surfaces in Spherical Coord.



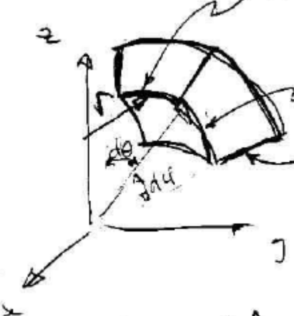
sphere $\rho = c$ φ, θ vary



cone $\varphi = c$ ρ, θ vary

plane $\theta = c$ ρ, φ vary

Elemental volume



edge lengths

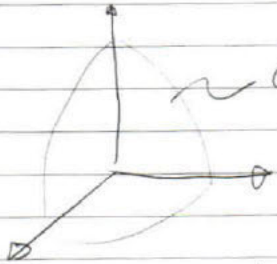
Elemental surfaces

Volume $$dV = (d\rho)(\rho\, d\varphi)(\rho \sin\varphi\, d\theta)$$
$$= \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

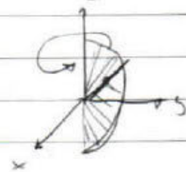
(10.7)

(8)

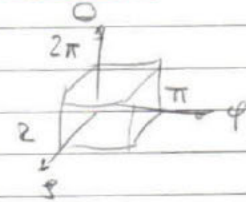
Ex



Calculate volume of sphere of radius R



$$V = \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi}\int_{\rho=0}^{R} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$



$$= \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} \frac{R^3}{3} \sin\varphi \, d\varphi \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{R^3}{3}\left(-\cos\pi + \cos 0\right) d\theta \quad = \int_{\theta=0}^{2\pi} \frac{2R^3}{3} \, d\theta$$

$$= \frac{2}{3}R^3 (2\pi) = \frac{4}{3}\pi R^3$$

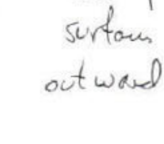
θ, φ, ρ

Now in order

$$V = \int_{\rho=0}^{R} \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} \rho^2 \sin\varphi \, \underline{d\varphi \, d\theta \, d\rho}$$

$$= \int_{\rho=0}^{R} \int_{\theta=0}^{2\pi} 2\rho^2 \, d\theta \, d\rho$$

$$= \int_{\rho=0}^{R} \underbrace{2\rho^2 \, 2\pi}_{\text{spherical surface}} \, d\rho$$



sweep spherical surface $4\pi r^2$ outward

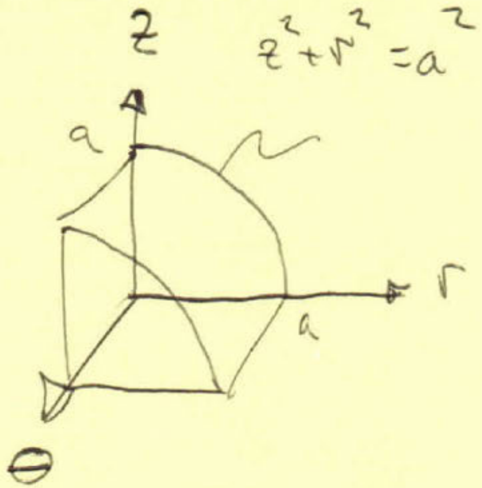
$$= 4\pi \left. \frac{\rho^3}{3} \right|_0^R$$

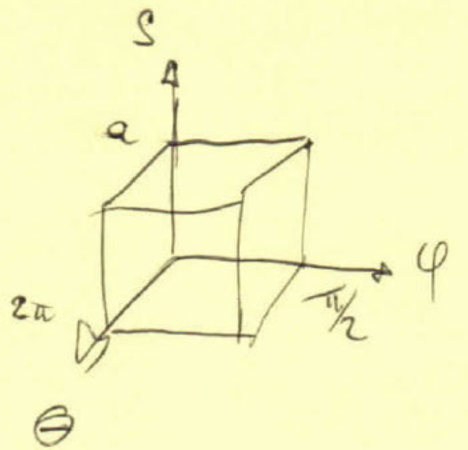
$$= \frac{4}{3}\pi R^3$$

This is why $\frac{d}{dr}(\text{Volume}) = \text{surface}$

$$\frac{d}{dr}\left(\frac{4}{3}\pi R^3\right) = 4\pi R^2$$

10.7



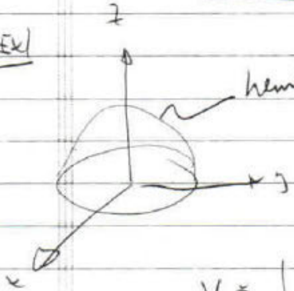


§13.6 Volume + Triple Integral

Cyl. + Sph Coord

1

Ex1



hemisphere $\underbrace{x^2+y^2+z^2}_{\rho^2} = a^2$

$\rho^2 = a^2$

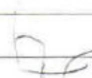
$\boxed{\text{Spherical Coord}}$

$$V = \int \int \int (d\rho)(\rho\, d\varphi)(\rho \sin\varphi\, d\theta)$$

$$= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/2} \int_{\rho=0}^{a} \rho^2 \sin\varphi \, \underline{d\rho\, d\varphi\, d\theta}$$

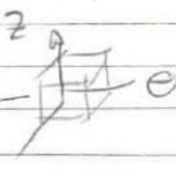
$$= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/2} \frac{a^3}{3} \sin\varphi \, d\varphi \quad d\theta$$

$$-\cos\varphi \Big|_0^{\pi/2}$$

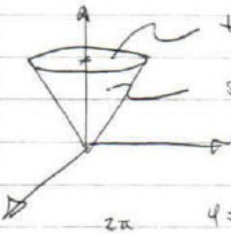


$$= \int_{\theta=0}^{2\pi} \frac{a^3}{3} \cdot 1 \quad d\theta$$

$$= 2\pi \frac{a^3}{3} = \frac{2\pi}{3} a^3 \quad \text{as expected.}$$



Spherical



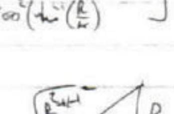
top $z=H$ $\Rightarrow$ $\rho\cos\varphi = H$

side $r=\left(\frac{R}{H}\right)z$ $\Rightarrow$ $\rho\sin\varphi = \left(\frac{R}{H}\right)\rho\cos\varphi$

$$V = \int_{\theta=0}^{2\pi}\int_{\varphi=\tan^{-1}\left(\frac{R}{H}\right)}^{\varphi=0}\int_{\rho=0}^{\rho=H/\cos\varphi} \rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\tan^{-1}(R/H)} \frac{H^3}{3\cos^3\varphi}\cdot\sin\varphi \, d\varphi \, d\theta$$

$$\frac{H^3}{2\cdot 3}\frac{1}{\cos^2\varphi}\Bigg|_0^{\tan^{-1}(R/H)} = \frac{H^3}{6}\left[\frac{1}{\cos^2\left(\tan^{-1}\left(\frac{R}{H}\right)\right)} - 1\right]$$



$$\cos\varphi = \frac{H}{\sqrt{R^2+H^2}}$$

$$\frac{1}{\cos^2\left(\tan^{-1}\left(\frac{R}{H}\right)\right)} - 1$$

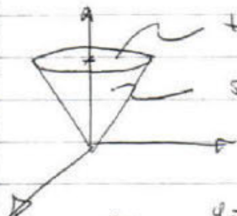
$$= \frac{R^2+H^2}{H^2} - 1 = \frac{R^2}{H^2}$$

$$= \int_{\theta=0}^{2\pi} \frac{H^3}{6}\cdot\frac{R^2}{H^2}\, d\theta$$

$$= \frac{HR^2}{6}2\pi$$

$$= \frac{HR^2\pi}{3}$$

Spherical



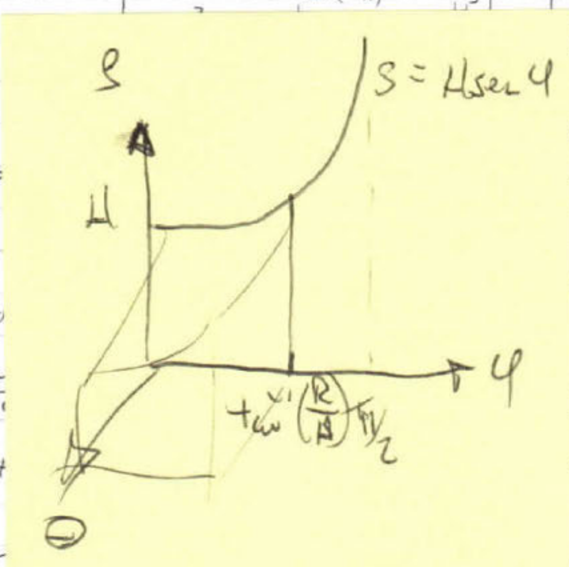
top $z = H$ $\Rightarrow \rho\cos\varphi = H$

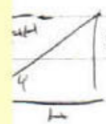
side $r = \left(\frac{R}{H}\right) z$ $\Rightarrow \rho\sin\varphi = \left(\frac{R}{H}\right)\rho\cos\varphi$

$$V = \int_{\theta=0}^{2\pi} \int_{\varphi=\tan^{-1}\left(\frac{R}{H}\right)}^{\varphi=0} \int_{\rho=0}^{\rho=H/\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\tan^{-1}(R/H)} \frac{H^3}{3\cos^3\varphi} \cdot \sin\varphi \, d\varphi \, d\theta$$

$\tan^{-1}(R/H)$ $H^3 \Big[\ldots \left(\frac{R}{H}\right) - 1 \Big]$





2π

$= \int$

$\theta = 0$

$= \frac{H}{\sqrt{R^2+H^2}}$

-1

$= \frac{R^2}{H^2}$

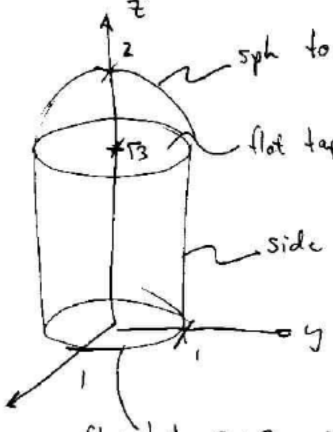
13.6

2

#11 or

$$V = \int_{r=0}^{1}\int_{z=0}^{z=\sqrt{4-r^2}}\int_{\theta=0}^{2\pi} 1\cdot r\,(d\theta)(dz)\,dr$$

#31 Same problem but in spherical coord.



sph top $x^2+y^2+z^2=2^2 \Rightarrow \boxed{\rho^2=2^2}$ top

flat top $z=\sqrt{3} \Rightarrow \boxed{\rho\cos\varphi=\sqrt{3}}$

side $x^2+y^2=1 \Rightarrow r^2=1^2$

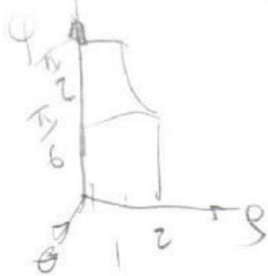
$\Rightarrow (\rho\sin\varphi)^2=1^2$

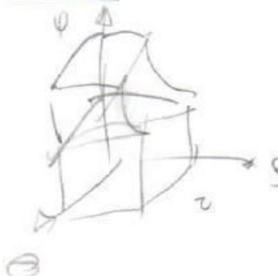
$\Rightarrow \boxed{\rho\sin\varphi=1}$ side

flat bot $z=0 \Rightarrow \boxed{\rho\cos\varphi=0} \Rightarrow \varphi=\pi/2$ bottom

$$V = \iiint 1\cdot \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \iiint 1\cdot \underbrace{(d\rho)(\rho\, d\varphi)(\rho\sin\varphi\, d\theta)}_{dV}$$





Spherical #31 (3)

$$V = \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi/6}\int_{\rho=0}^{2} 1\cdot \rho^2 \sin\varphi\,(d\rho)(d\varphi)(d\theta)$$

$$+\int_{\theta=0}^{2\pi}\int_{\varphi=\pi/6}^{\pi/2}\int_{\rho=0}^{\rho=1/\sin\varphi} 1\cdot \rho^2 \sin\varphi\,(d\rho)(d\varphi)(d\theta)$$ empty can

or

$$V = \int_{\theta=0}^{2\pi}\int_{\rho=\phi}^{2}\int_{\varphi=\pi/6}^{\varphi=\sin^{-1}(1/\rho)} \rho^2 \sin\varphi\; d\varphi\; d\rho\; d\theta$$ empty cone

$$+\int_{\theta=0}^{2\pi}\int_{\rho=0}^{1}\int_{\varphi=\pi/6}^{\varphi=\pi/2} \rho^2 \sin\varphi\; d\varphi\; d\rho\; d\theta$$ hemi sphere at bot

$$+\int_{\theta=0}^{2\pi}\int_{\rho=0}^{2}\int_{\varphi=0}^{\pi/6} \rho^2 \sin\varphi\; d\varphi\; d\rho\; d\theta$$

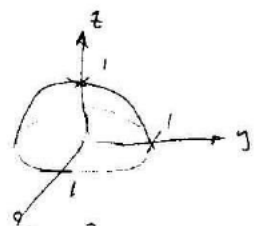
13.6

4

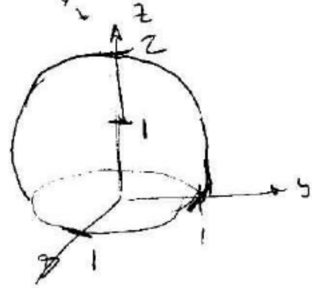
#34

bottom

hemisphere $\rho = 1 \quad z \geq 0$



top : cardioid of rev $\rho = 1 + \cos\varphi$



spherical coord:

$$V = \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/2} \int_{\rho=1}^{\rho=1+\cos\varphi} \rho^2 \sin\varphi \, (d\rho)(d\varphi)(d\theta)$$

$$= \frac{1}{3} \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/2} \left((1+\cos\varphi)^3 - 1^3 \right) \sin\varphi \; d\varphi \, d\theta$$

$$= \frac{1}{3} \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/2} \left(3\cos\varphi + 3\cos^2\varphi + \cos^3\varphi \right) \sin\varphi \, d\varphi \, d\theta$$

5

$$V = \frac{1}{3} \int_{\theta=0}^{2\pi} \left(-\frac{3}{2}\cos^2\varphi - \cos^3\varphi - \frac{1}{4}\cos^4\varphi \Big|_{\varphi=0}^{\pi/2} \right) d\theta$$

$$= \frac{1}{3}\int_{\theta=0}^{2\pi} \left(\frac{3}{2} + 1 + \frac{1}{4} \right) d\theta = \frac{11}{3\cdot 4} \int_{\theta=0}^{2\pi} d\theta$$

$$V = \frac{11\pi}{6}$$

Sph Cyl

$$V = \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/2} \int_{\rho=1}^{(1+\cos\varphi)\sin\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$